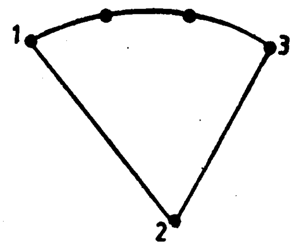

Fig. 8.8(e) Triangular element with curved side as cubic polynomial

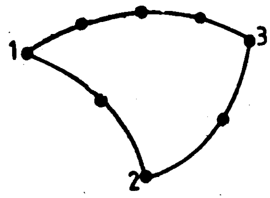

Fig. 8.8(f) Triangular element with curved sides

The parametric equation to the curved side 13 becomes

$$x = x_1 + (x_3 - x_1)\eta + \eta(1-\eta)(\alpha_1 + \alpha_2\eta)$$
$$y = y_1 + (y_3 - y_1)\eta + \eta(1-\eta)(\beta_1 + \beta_2\eta) \qquad (8.134)$$

which may be reduced to a parabola by choosing

$$x_5 = x_4 - \tfrac{1}{3}(x_1 - x_3)$$
$$y_5 = y_4 - \tfrac{1}{3}(y_1 - y_3)$$

Triangle with curved sides

We consider a curved triangular element 123 as shown in Figure 8.8(f). We take the quadratic polynomial on two sides and fourth degree Lagrange interpolation on the third side.

The isoparametric mapping is given by

$$x = x_1 + (x_2 - x_1)\xi + (x_3 - x_1)\eta + \eta(1 - \xi - \eta)(\alpha_1 + \alpha_2\eta + \alpha_3\eta^2)$$
$$y = y_1 + (y_2 - y_1)\xi + (y_3 - y_1)\eta + \eta(1 - \xi - \eta)(\beta_1 + \beta_2\eta + \beta_3\eta^2) \qquad (8.135)$$

where

$$\alpha_1 = -\frac{22}{3}x_1 - 2x_3 + \frac{16}{3}x_4 - 12x_5 + 16x_6$$

$$\alpha_2 = 16x_1 + \frac{16}{3}x_3 - 32x_4 + 64x_5 - \frac{160}{3}x_6$$

$$\alpha_3 = -\frac{32}{3}x_1 - \frac{32}{3}x_3 + \frac{128}{3}x_4 - 64x_5 + \frac{128}{3}x_6$$

$$\beta_1 = -\frac{22}{3}y_1 - 2y_3 + \frac{16}{3}y_4 - 12y_5 + 16y_6$$

$$\beta_2 = 16y_1 + \frac{16}{3}y_3 - 32y_4 + 64y_5 - \frac{160}{3}y_6$$

$$\beta_3 = -\frac{32}{3}y_1 - \frac{32}{3}y_3 + \frac{128}{3}y_4 - 64y_5 + \frac{128}{3}y_6$$

The parametric equation to the curved side becomes

$$x = x_1 + (x_3 - x_1)\eta + \eta(1-\eta)(\alpha_1 + \alpha_2\eta + \alpha_3\eta^2)$$
$$y = y_1 + (y_3 - y_1)\eta + \eta(1-\eta)(\beta_1 + \beta_2\eta + \beta_3\eta^2) \quad (8.136)$$

8.4.8 Numerical integration over finite element

The formulae for evaluating integrals over the elements in one, two and three dimensions are given by (8.54), (8.72) and (8.111) respectively when the integrand is a polynomial integral powers of the local coordinates. We now evaluate the integral of the form

$$I = \int_{(e)} F(\mathbf{x})\, d\mathbf{x} \quad (8.137)$$

where $F(\mathbf{x})$ is a given function, (e) is the element and $\mathbf{x}$ represents one or multidimensional coordinates. The integral (8.137) in terms of the local coordinates becomes

$$I = \int_{(S)} |\mathbf{J}|\, f(\mathbf{L})\, d\mathbf{L} \quad (8.138)$$

where (S) is the standard element, $\mathbf{L}$ are the local coordinates, $|\mathbf{J}|$ is the Jacobian and f is the transformed F. The integral (8.138) after neglecting the truncation error term may be replaced by the formula

$$\int_{(S)} |J|\, f(\mathbf{L})\, d\mathbf{L} = \sum_{i=1}^{n} W_i f(\mathbf{L}^{(i)}) \quad (8.139)$$

where W_i and $\mathbf{L}^{(i)}$ are the weight coefficients and the abscissas respectively.

We discuss the quadrature formulas for the line segment and the triangular elements.

Line segment element

The interval $[x_i, x_{i+1}]$ is the line segment element and the local coordinate is defined by (8.51),

$$L = \frac{2x - (x_{i+1} + x_i)}{(x_{i+1} - x_i)}$$

The quadrature formula (8.139) becomes

$$I = \frac{(x_{i+1} - x_i)}{2} \int_{-1}^{1} f(L)\, dL = \frac{l^{(e)}}{2} \sum_{i=1}^{n} W_i^{(e)} f(L_i^{(e)}) \quad (8.140)$$

where $l^{(e)} = x_{i+1} - x_i$ is the length of the element (e). The values of $W_i^{(e)}$ and $L^{(e)}$ for the Gauss quadrature formula are listed in Table 8.1.

TABLE 8.1 ABSCISSAS AND WEIGHTS FOR GAUSS QUADRATURE FORMULAS

n	$\pm L_i^{(e)}$			$W_i^{(e)}$		
1	0			2		
2	$1/\sqrt{3}$			1		
3	0			8/9		
	$\sqrt{3/5}$			5/9		
4	$((15-2\sqrt{30})/35)^{1/2}$			0.65214	51548	62546
	$((15+2\sqrt{30})/35)^{1/2}$			0.34785	48451	37454
5	0			0.56888	88888	88889
	$((35-2\sqrt{70})/63)^{1/2}$			0.47862	86704	99366
	$((35+2\sqrt{70})/63)^{1/2}$			0.23692	68850	56189
6	0.23861	91860	83197	0.46791	39345	72691
	0.66120	93864	66265	0.36076	15730	48139
	0.93246	95142	03152	0.17132	44923	79170

Triangular element

The formula (8.139) becomes

$$I = 2\Delta^{(e)} \int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)\, dL_1\, dL_2$$

$$= 2\Delta^{(e)} \sum_{i=1}^{n} W_i^{(e)} f(L_1^{(e)}, L_2^{(e)}, L_3^{(e)}) \qquad (8.141)$$

where $\Delta^{(e)}$ is the area of the triangular element. The values of the local co-ordinates $L_i^{(e)}$ and the weights $W_i^{(e)}$ are given in Table 8.2.

TABLE 8.2 ABSCISSAS AND WEIGHTS FOR QUADRATURE FORMULAS FOR TRIANGLES

n	i	$L_1^{(e)}$	$L_2^{(e)}$	$L_3^{(e)}$	$W_i^{(e)}$
1 (Linear)	1	1/3	1/3	1/3	1/2
2 (quadratic)	1	1/2	1/2	0	1/6
	2	0	1/2	1/2	1/6
	3	1/2	0	1/2	1/6
3 (cubic)	1	1/3	1/3	1/3	−9/32
	2	3/5	1/5	1/5	25/96
	3	1/5	3/5	1/5	25/96
	4	1/5	1/5	3/5	25/96

8.5 FINITE ELEMENT METHODS

For the sake of simplicity, let us limit equations (8.1) and (8.2) to a two-dimensional boundary value problem

$$L[u] = r(x, y), \qquad (x, y) \in \mathfrak{R} \qquad (8.142)$$

$$U_\mu[u] = r_\mu, \qquad (x, y) \in \partial\mathfrak{R} \qquad (8.143)$$

where $u(x, y)$ is the unknown function, L denotes the differential operator with the highest order derivative m, r and r_μ are given functions, and U_μ represents the boundary differential operator.

We divide the domain $\mathcal{R}$ into M finite elements $\mathcal{R}^{(1)}, \mathcal{R}^{(2)}, \ldots, \mathcal{R}^{(M)}$ and write the approximate solution w in (8.4) as a sum of the piecewise approximate solution as

$$w = u^{(1)} + u^{(2)} + \ldots + u^{(e)} + \ldots + u^{(M)} \tag{8.144}$$

where the superscript (e) denotes the eth element and $u^{(e)}$ is only defined in the element (e) and is taken to be zero elsewhere. The function $u^{(e)}$ is generally expressed in terms of the shape functions and nodal parameters of the element. We have

$$u^{(e)} = \mathbf{N}^{(e)}\boldsymbol{\phi}^{(e)} \tag{8.145}$$

where $\mathbf{N}^{(e)}$ is the row vector of the shape functions and $\boldsymbol{\phi}^{(e)}$ is the column vector which depends on the nodal values of the function u or its derivatives.

Further we can also write the function $u^{(e)}$ in the form

$$u^{(e)} = \bar{\mathbf{N}}^{(e)}\boldsymbol{\phi} \tag{8.146}$$

where $\bar{\mathbf{N}}^{(e)}$ is an extended $1 \times N$ row vector, $\boldsymbol{\phi} = [\phi_1\phi_2\ldots\phi_N]^T$ is $N \times 1$ column vector and N denotes the total number of nodal parameters in the domain $\mathcal{R}$. The approximate solution w in (8.144) becomes

$$w = \sum_{e=1}^{M} \bar{\mathbf{N}}^{(e)}\boldsymbol{\phi} \tag{8.147}$$

Substituting (8.145) into (8.142), we write the residue in the differential equation within the finite element $\mathcal{R}^{(e)}$ as

$$E^{(e)}[u^{(e)}] = L[u^{(e)}] - r(x, y) \tag{8.148}$$

The residue in the differential equation (8.142) in the domain $\mathcal{R}$ is given by

$$E[w] = L[w] - r(x, y) \tag{8.149}$$

The nodal parameters ϕ may be determined by using any one of the methods in Subsections 8.2.1-8.2.5 and 8.3.1.

8.5.1 Ritz finite element method

Let us assume that there exists a variational principle for the boundary value problem (8.142)-(8.143) and that we look for an extremum of a function of the form

$$J[u] = \int_{\mathcal{R}} F \, d\mathcal{R} + \Psi \tag{8.150}$$

where Ψ may be zero or an integral over parts of the boundary $\partial\mathcal{R}$. Substituting the approximate solution w from (8.147) into (8.150) and extremizing the function $J[u]$ with respect to the nodal parameters $\boldsymbol{\phi}$, we obtain the equations

$$\frac{\partial J[w]}{\partial \boldsymbol{\phi}}=\left[\frac{\partial J}{\partial \phi_1}\frac{\partial J}{\partial \phi_2}\cdots\frac{\partial J}{\partial \phi_N}\right]^T=0 \tag{8.151}$$

If we assume that the functional $J[w]$ can be written as a sum of element contributions as

$$J[w]=\sum_{e=1}^{M} J^{(e)} \tag{8.152}$$

where the quantity $J^{(e)}$ may be called the *element functional* then the equations (8.152) become

$$\frac{\partial J}{\partial \phi_i}=\sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial \phi_i}=0,\ i=1(1)N \tag{8.153}$$

The summation is over all the elements in the domain $\mathcal{R}$ and node i may be common to several finite elements. The equation (8.153) gives the finite element discretization of the differential equation (8.142) at the nodes $i=1(1)N$. Incorporating the boundary conditions we get a system of equations which may be solved by the methods discussed in Section 7.4.

Alternatively we may consider the extremization of the element functional $J^{(e)}$ with respect to the element nodal parameters $\boldsymbol{\phi}^{(e)}$ and obtain

$$\frac{\partial J^{(e)}}{\partial \boldsymbol{\phi}^{(e)}}=0 \tag{8.154}$$

The equation (8.154) gives the required finite element equations for a typical element (e). These element equations are assembled according to (8.153) to obtain the overall equations.

8.5.2 Least square finite element method

Using (8.149), the square of the residual E over the entire domain $\mathcal{R}$ becomes

$$WE[w]=\iint_{\mathcal{R}} E^2\, d\mathcal{R} \tag{8.155}$$

The necessary conditions for $WE[w]$ to be minimum are given by

$$\frac{\partial WE}{\partial \phi_i}=2\iint_{\mathcal{R}} E\,\frac{\partial E}{\partial \phi_i}\, d\mathcal{R}=0,\ i=1(1)N \tag{8.156}$$

which gives the required system of equations.

Assuming that (8.155) can be expressed as a sum of element integrals and then substituting (8.148), the equations (8.156) become

$$\frac{\partial WE}{\partial \phi_i}=\sum_{e=1}^{M}2\iint_{\mathcal{R}^{(e)}} E^{(e)}\,\frac{\partial E^{(e)}}{\partial \phi_i}\, d\mathcal{R}^{(e)}=0,\ i=1(1)N \tag{8.157}$$

Summing over all the elements of the domain $\mathcal{R}$ we obtain the least square finite element discretization of the differential equation (8.142) at the node i. The least square finite element equation for a typical element (e) may be written as

$$\frac{\partial WE^{(e)}}{\partial \phi^{(e)}}=2\iint_{\mathcal{R}^{(e)}} E^{(e)}\frac{\partial E^{(e)}}{\partial \phi^{(e)}}\,d\mathcal{R}^{(e)}=0 \tag{8.158}$$

Assembling the element equations as in (8.157), we obtain the matrix equation for the nodal parameters ϕ.

8.5.3 Galerkin finite element method

Using (8.147) and (8.149) the Galerkin system of equations (8.16) become

$$\iint_{\mathcal{R}} (\sum_{e=1}^{M} \bar{N}^{(e)})E[w]\,dx\,dy=0 \tag{8.159}$$

which may be written as

$$\sum_{e=1}^{M} (\iint_{\mathcal{R}^{(e)}} \bar{N}^{(e)}E[w])\,dx\,dy=0 \tag{8.160}$$

The equation (8.160) represents the system of equations in the nodal parameters.

The Galerkin equations for a typical element (e) are given by

$$\iint_{\mathcal{R}^{(e)}} N^{(e)T}E^{(e)}[u^{(e)}]\,dx\,dy=0 \tag{8.161}$$

The summation of the element equations (8.161) according to (8.160) gives the matrix equation for the nodal parameters ϕ.

We may again show as in (8.43) that the Ritz finite element equations (8.153) and the Galerkin finite element equations (8.160) are identical matrix equations.

8.5.4 Convergence analysis

The accuracy in the finite element solution can be increased either by decreasing the size of the elements or by increasing the degree of the polynomial in the piecewise approximate solution. The convergence of the finite element solution to the exact solution as the size of the finite element approaches zero is obtained if the following conditions are satisfied:

Completeness

This is the condition that, as the size of finite element approaches zero, the terms occurring under the integral sign in the weighted residual or variational formulation must tend to be single valued and well behaved.

Thus the set of shape functions chosen must be able to represent any constant value of the function u as well as the derivatives upto order m (highest derivatives occurring in WE or $J[u]$) within each element in the limit as element size approaches zero.

Compatibility

This is an interelement continuity condition. If the order of the highest derivative in the weighted residual equation WE (or functional $J[u]$) is m, then the finite elements and the shape functions are to be selected such that at the element interfaces, the function u has continuity of all the derivatives upto the order $m-1$.

The finite elements satisfying this criterion are called the *conforming elements,* otherwise *nonconforming elements.* If the compatibility condition is satisfied then it is possible to express WE or $J[u]$ as a sum of elemental contributions.

8.6 BOUNDARY VALUE PROBLEMS IN ORDINARY DIFFERENTIAL EQUATIONS

We solve the linear boundary value problem

$$-\frac{d}{dx}\left(p(x)\frac{du}{dx}\right)+q(x)u=r(x) \tag{8.162}$$

subject to the boundary conditions

$$u(a)=\gamma_1,\ u(b)=\gamma_2 \tag{8.163}$$

Let us consider the functional

$$J[u]=\frac{1}{2}\int_a^b\left[p\left(\frac{du}{dx}\right)^2+qu^2-2ru\right]dx \tag{8.164}$$

It is easily verified that the necessary condition for $\delta J[u]=0$ is given by (8.162). We discretize the interval $[a, b]$ with $N+2$ nodes and $N+1$ elements as shown in Figure 8.2. We assume that the functional J can be written as a sum of $N+1$ elemental quantities $J^{(e)}$ as

$$J[u]=\sum_{e=1}^{N+1} J^{(e)} \tag{8.165}$$

where

$$J^{(e)}=\frac{1}{2}\int_{x_j}^{x_k}\left[p\left(\frac{du^{(e)}}{dx}\right)^2+q(u^{(e)})^2-2ru^{(e)}\right]dx \tag{8.166}$$

and, x_j and x_k are the coordinates of the end nodes of a typical line segment element (e) as shown in Figure 8.9. The function $u^{(e)}$ is defined over the element (e) and zero elsewhere. We take the approximate solution in the form (see (8.45))

$$u^{(e)}=N_ju_j+N_ku_k=\mathbf{N}^{(e)}\boldsymbol{\phi}^{(e)} \tag{8.167}$$

$$+\dots\begin{bmatrix} a^{(i)}_{i-1,i-1} & a^{(i)}_{i-1,i} \\ a^{(i)}_{i,i-1} & a^{(i)}_{i,i}\end{bmatrix}\begin{bmatrix} u_{i-1} \\ u_i\end{bmatrix}+\begin{bmatrix} a^{(i+1)}_{i,i} & a^{(i+1)}_{i,i+1} \\ a^{(i+1)}_{i+1,i} & a^{(i+1)}_{i+1,i+1}\end{bmatrix}\begin{bmatrix} u_i \\ u_{i+1}\end{bmatrix}$$

$$+\dots+\begin{bmatrix} a^{(N+1)}_{N,N} & a^{(N+1)}_{N,N+1} \\ a^{(N+1)}_{N+1,N} & a^{(N+1)}_{N+1,N+1}\end{bmatrix}\begin{bmatrix} u_N \\ u_{N+1}\end{bmatrix}-$$

$$\begin{bmatrix} b^{(1)}_0 \\ b^{(1)}_1\end{bmatrix}-\begin{bmatrix} b^{(2)}_1 \\ b^{(2)}_2\end{bmatrix}\dots-\begin{bmatrix} b^{(i)}_{i-1} \\ b^{(i)}_i\end{bmatrix}\dots-\begin{bmatrix} b^{(N+1)}_N \\ b^{(N+1)}_{N+1}\end{bmatrix}=0 \tag{8.177}$$

The matrix equation resulting from the assembling of the element equations from the ith and $(i+1)$th elements is obtained by adding the equations in rows 2 and 3. The assembled matrix equation becomes

$$\begin{array}{ccc} u_{i-1} & u_i & u_{i+1} \end{array}$$

$$\begin{bmatrix} a^{(i)}_{i-1,i-1} & a^{(i)}_{i-1,i} & 0 \\ a^{(i)}_{i,i-1} & a^{(i)}_{i,i}+a^{(i+1)}_{i,i} & a^{(i+1)}_{i,i+1} \\ 0 & a^{(i+1)}_{i+1,i} & a^{(i+1)}_{i+1,i+1}\end{bmatrix}\begin{bmatrix} u_{i-1} \\ u_i \\ u_{i+1}\end{bmatrix} = \begin{bmatrix} b^{(i)}_{i-1} \\ b^{(i)}_i + b^{(i+1)}_i \\ b^{(i+1)}_{i+1}\end{bmatrix} \tag{8.178}$$

We apply this process to $N+1$ elements (see Fig. 8.2) and get single matrix equation

$$\mathbf{A}\boldsymbol{\phi}=\mathbf{b} \tag{8.179}$$

where $\mathbf{A}$ and $\mathbf{b}$ are $(N+2)\times(N+2)$ and $(N+2)\times 1$ matrices respectively. The equation (8.179) is represented in schematic form in Figure 8.10.

The finite element discretization of the differential equation (8.162) at $x=x_i$ is given by the $(i+1)$th row of the equation (8.179):

$$-\left(\frac{p^{(i)}}{l^{(i)}}-\frac{q^{(i)}}{6}l^{(i)}\right)u_{i-1}+\left(\frac{p^{(i)}}{l^{(i)}}+\frac{p^{(i+1)}}{l^{(i+1)}}+\frac{q^{(i)}}{3}l^{(i)}+\frac{q^{(i+1)}}{3}l^{(i+1)}\right)u_i$$

$$-\left(\frac{p^{(i+1)}}{l^{(i+1)}}-\frac{q^{(i+1)}}{6}l^{(i+1)}\right)u_{i+1}=\frac{1}{2}(r^{(i)}l^{(i)}+r^{(i+1)}l^{(i+1)}) \tag{8.180}$$

The equation (8.180) for $i=1(1)N$ leads to a set of N equations in $N+2$ unknowns, $u_0, u_1, \dots, u_{N+1}$. Substituting the boundary conditions $u_0=\gamma_1$, $u_{N+1}=\gamma_2$, we get a tridiagonal system of N equations in N unknowns $u_1, u_2, \dots, u_N$.

Alternatively we may incorporate the boundary conditions (8.163) into (8.179) directly. For example, to incorporate the condition $y_0 = \gamma_1$ into (8.179) we may adopt the following procedure:

(i) Multiply all the off-diagonal elements of the first column in **A** by γ_1 and transfer them to the right side of the equation (8.179).
(ii) Set all the off-diagonal elements in the first column in **A** to zero.
(iii) Set in **A** all the off-diagonal elements of the first row to zero and the diagonal element equal to one.
(iv) Set the first element in **b** to γ_1.

In a similar manner we may incorporate the condition $u_{N+1} = \gamma_2$ into (8.179). The system of equations (8.179) incorporating the conditions (8.163) can be written in schematic form as

$$\begin{bmatrix} 1 & \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \times & \times & \cdot & \cdots & \cdot \\ \cdot & \times & \times & \times & \cdots & \cdot \\ \vdots & & & & & \\ \cdot & & & \times & \times & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{bmatrix} \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_N \\ u_{N+1} \end{bmatrix} = \begin{bmatrix} \gamma_1 \\ \times \\ \times \\ \times \\ \times \\ \times \\ \times \\ \gamma_2 \end{bmatrix} - \gamma_1 \begin{bmatrix} \cdot \\ \times \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \end{bmatrix} - \gamma_2 \begin{bmatrix} \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \times \\ \cdot \end{bmatrix} \tag{8.181}$$

where ($\times$) and ($\cdot$) denote a number and an empty location respectively. The matrix **A** in (8.181) is a tridiagonal symmetric matrix.

8.6.2 Mixed boundary conditions

If we associate with the differential equation (8.162), the mixed boundary conditions

$$\alpha_0 u(a) + \alpha_1 u'(a) = \gamma_1$$
$$\beta_0 u(b) + \beta_1 u'(b) = \gamma_2 \tag{8.182}$$

then, instead of (8.164) we use the functional

$$J[u] = \frac{1}{2}\int_a^b (pu'^2 + qu^2 - 2ru)\, dx + \frac{p(a)}{2\alpha_1}(-\alpha_0 u^2(a) + 2\gamma_1 u(a))$$
$$+ \frac{p(b)}{2\beta_1}(\beta_0 u^2 u(b) - 2\gamma_2 u(b)) \tag{8.183}$$

The first and the last element functionals respectively may be written as

$$J^{(1)} = \frac{1}{2}\int_{x_0}^{x_1} [p(u^{(e)'})^2 + q(u^{(e)})^2 - 2ru^{(e)}]\, dx + \frac{p_0}{2\alpha_1}(-\alpha_0 u_0^2 + 2\gamma_1 u_0)$$

or

$$\left\{-\begin{bmatrix} N_j \\ N_k \end{bmatrix} p(x) \frac{du^{(e)}}{dx}\right\}_{x_j}^{x_k} + \mathbf{A}^{(e)}\boldsymbol{\phi}^{(e)} - \mathbf{b}^{(e)} = 0 \tag{8.189}$$

The term $\left\{-\begin{bmatrix} N_j \\ N_k \end{bmatrix} p(x) \frac{du^{(e)}}{dx}\right\}_{x_j}^{x_k}$ contributes to the terms of $\mathbf{b}^{(e)}$ if the derivative du/dx is specified at either end of the element (e) otherwise it is neglected. Summing (8.189) for all the elements we get the system of equations (8.179) which is the same as obtained by using the Ritz method.

Example 8.6 Solve by the finite element method the boundary value problem

$$u'' + (1 + x^2)u + 1 = 0, \; u(\pm 1) = 0$$

Using (8.164), the functional can be written as

$$J[u] = \frac{1}{2}\int_{-1}^{1} (-u'^2 + (1 + x^2)u^2 + 2u)\, dx$$

We have

$$\mathbf{A}^{(e)} = \int_{x_j}^{x_k} \left\{ -\begin{bmatrix} N_j'N_j' & N_j'N_k' \\ N_k'N_j' & N_k'N_k' \end{bmatrix} + (1 + x^2)\begin{bmatrix} N_jN_j & N_jN_k \\ N_kN_j & N_kN_k \end{bmatrix} \right\} dx$$

$$\mathbf{b}^{(e)} = \int_{x_j}^{x_k} \begin{bmatrix} N_j \\ N_k \end{bmatrix} dx, \quad \boldsymbol{\phi}^{(e)} = \begin{bmatrix} u_j \\ u_k \end{bmatrix}, \quad N_j = \frac{1}{l^{(e)}}(x_k - x)$$

$$N_k = \frac{1}{l^{(e)}}(x - x_j), \quad l^{(e)} = x_k - x_j$$

Simplifying we get

$$\mathbf{A}^{(e)} = -\frac{1}{l^{(e)}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{l^{(e)}}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$+ \frac{l^{(e)}}{60}\begin{bmatrix} 2(10x_j^2 + 5x_jl^{(e)} + l^{(e)2}) & 10x_j^2 + 10x_jl^{(e)} + 3l^{(e)2} \\ 10x_j^2 + 10x_jl^{(e)} + 3l^{(e)2} & 2(10x_j^2 + 15x_jl^{(e)} + 6l^{(e)2}) \end{bmatrix}$$

$$\mathbf{b}^{(e)} = \frac{l^{(e)}}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

We now compute the results for the following cases:

(i) $h = 1$, there are two elements as shown in Fig. 8.11(a).

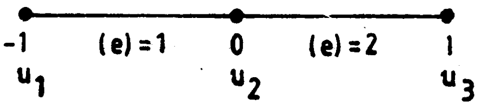

Fig. 8.11(a) Representation of elements

We have
for element (1);

$$x_j = -1,\ x_k = 0,\ l^{(1)} = 1,$$

$$A^{(1)} = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{60}\begin{bmatrix} 12 & 3 \\ 3 & 2 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} -28 & 73 \\ 73 & -38 \end{bmatrix}$$

$$b^{(1)} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for element (2);

$$x_j = 0,\ x_k = 1,\ l^{(2)} = 1,$$

$$A^{(2)} = -\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{1}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{1}{60}\begin{bmatrix} 2 & 3 \\ 3 & 12 \end{bmatrix}$$

$$= \frac{1}{60}\begin{bmatrix} -38 & 73 \\ 73 & -28 \end{bmatrix}$$

$$b^{(2)} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Collecting the contribution from both the elements (1) and (2) we obtain

$$\frac{1}{60}\begin{bmatrix} -28 & 73 & 0 \\ 73 & -(38+38) & 73 \\ 0 & 73 & -28 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ 1+1 \\ 1 \end{bmatrix} = 0$$

The boundary conditions, $u_1 = 0$, $u_3 = 0$, are incorporated by deleting the rows and columns corresponding to u_1 and u_3. We get the required equation

$$-\frac{76}{60}u_2 + 1 = 0$$

or $$u_2 = \frac{60}{76} = \frac{15}{19} = 0.78947$$

(ii) $h = \frac{1}{2}$, there are four elements as shown in Fig. 8.11(b)

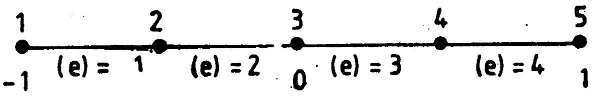

Fig. 8.11(b) Representation of elements

We have

for element (1); $x_j = -1,\ x_k = -\frac{1}{2},\ l^{(1)} = \frac{1}{2}$

$$A^{(1)} = \frac{1}{480}\begin{bmatrix} -818 & 1023 \\ 1023 & -848 \end{bmatrix},\ b^{(1)} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for element (2); $x_j = -\frac{1}{2},\ x_k = 0,\ l^{(2)} = \frac{1}{2}$

$$A^{(2)} = \frac{1}{480}\begin{bmatrix} -868 & 1003 \\ 1003 & -878 \end{bmatrix},\ b^{(2)} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for element (3); $x_j = 0,\ x_k = \frac{1}{2},\ l^{(3)} = \frac{1}{2}$

$$A^{(3)} = \frac{1}{480}\begin{bmatrix} -878 & 1003 \\ 1003 & -868 \end{bmatrix},\ b^{(3)} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for element (4); $x_j = \frac{1}{2},\ x_k = 1,\ l^{(4)} = \frac{1}{2}$

$$A^{(4)} = \frac{1}{480}\begin{bmatrix} -848 & 1023 \\ 1023 & -818 \end{bmatrix},\ b^{(4)} = \frac{1}{4}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Collecting the contribution from each element we have

$$\frac{1}{480}\begin{bmatrix} -818 & 1023 & 0 & 0 & 0 \\ 1023 & -1716 & 1003 & 0 & 0 \\ 0 & 1003 & -1756 & 1003 & 0 \\ 0 & 0 & 1003 & -1716 & 1023 \\ 0 & 0 & 0 & 1023 & -818 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} 1 \\ 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} = 0$$

Using the boundary conditions $u_1 = u_5 = 0$, we obtain the system of equations

$$\frac{1}{480}[-1716u_2 + 1003u_3] + \frac{1}{2} = 0$$

$$\frac{1}{480}[1003u_2 - 1756u_3 + 1003u_4] + \frac{1}{2} = 0$$

$$\frac{1}{480}[1003u_3 - 1716u_4] + \frac{1}{2} = 0$$

It is easily seen that $u_2 = u_4$ and we have the required two equations

$$\frac{1}{480}[-878u_3 + 1003u_4] + \frac{1}{4} = 0$$

$$\frac{1}{480}[1003u_3 - 1716u_4] + \frac{1}{2} = 0$$

Solving, we get

$$u_3 = 0.8921398, \; u_4 = 0.6613148$$

Example 8.7 Use the finite element Galerkin method to derive the difference schemes for the boundary value problem

$$u'' - Ku' = 0$$

$$u(0) = 1, \; u(1) = 0$$

where $K > 0$ is assumed constant.
Obtain the characteristic equation of the difference schemes.

The exact solution of the boundary value problem is

$$u = (e^{KX} - e^{K})/(1 - e^{K})$$

The Galerkin equations (8.187) for a typical line segment element (e) (Fig. 8.9) may be written as

$$\int_{x_j}^{x_k} \mathbf{N}^T \left(\frac{d^2u^{(e)}}{dx^2} - K\frac{du^{(e)}}{dx} \right) dx = 0$$

where $u^{(e)} = \mathbf{N}^{(e)}\boldsymbol{\phi}^{(e)}$ is the piecewise cubic Hermite polynomial,

$$\mathbf{N}^{(e)} = [N_j H_j N_k H_k], \; \boldsymbol{\phi}^{(e)} = [u_j \; u_j' \; u_k \; u_k']^T$$

and

$$N_j = L_j^2(3 - 2L_j)$$
$$H_j = l^{(e)}L_j^2L_k$$
$$N_k = L_k^2(3 - 2L_k)$$
$$H_k = -l^{(e)}L_jL_k^2$$

Using (8.54) the line segment element equation becomes

$$(\mathbf{A}^{(e)} - K\mathbf{B}^{(e)})\boldsymbol{\phi}^{(e)} = \mathbf{0}$$

where

$$\mathbf{A}^{(e)} = \frac{1}{30l^{(e)}}\begin{bmatrix} -36 & -3l^{(e)} & 36 & -3l^{(e)} \\ -3l^{(e)} & -4l^{(e)2} & 3l^{(e)} & l^{(e)2} \\ 36 & 3l^{(e)} & -36 & 3l^{(e)} \\ -3l^{(e)} & l^{(e)2} & 3l^{(e)} & -4l^{(e)2} \end{bmatrix}$$

is the contribution of the element (e) to the functional J. The conditions for minimization of the functional J in (8.196) with respect to the nodal values ϕ_i, $i=1(1)\,N$ give the following system of equations

$$\frac{\partial J}{\partial \phi_i}=\sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial \phi_i}=0,\quad i=1(1)N$$

or

$$\frac{\partial J}{\partial \boldsymbol{\phi}}=\sum_{e=1}^{M}\frac{\partial J^{(e)}}{\partial \boldsymbol{\phi}^{(e)}}=0 \tag{8.198}$$

since $J^{(e)}$ depends on the nodal values associated with the element (e) only.

The equation $\dfrac{\partial J^{(e)}}{\partial \boldsymbol{\phi}^{(e)}}=0$ is called the element equation. It generally turns out that one term of the summation gives the form for the other terms. Therefore, it is sufficient to explicitly consider the contribution of only a typical finite element (e). Differentiating (8.197) with respect to $\boldsymbol{\phi}^{(e)}$ we get

$$\frac{\partial J^{(e)}}{\partial \boldsymbol{\phi}^{(e)}}=\iint_{(e)}\left\{p\left(\frac{\partial \mathbf{N}^{(e)T}}{\partial x}\frac{\partial \mathbf{N}^{(e)}}{\partial x}+\frac{\partial \mathbf{N}^{(e)T}}{\partial y}\frac{\partial \mathbf{N}^{(e)}}{\partial y}\right)\boldsymbol{\phi}^{(e)}-r\,\mathbf{N}^{(e)T}\right\}dx\,dy \tag{8.199}$$

Thus, the element equations become

$$\mathbf{A}^{(e)}\boldsymbol{\phi}^{(e)}-\mathbf{b}^{(e)}=\mathbf{0} \tag{8.200}$$

where

$$\mathbf{A}^{(e)}=\iint_{(e)}p\left(\frac{\partial \mathbf{N}^{(e)T}}{\partial x}\frac{\partial \mathbf{N}^{(e)}}{\partial x}+\frac{\partial \mathbf{N}^{(e)T}}{\partial y}\frac{\partial \mathbf{N}^{(e)}}{\partial y}\right)dx\,dy$$

$$\mathbf{b}^{(e)}=\iint_{(e)}r\,\mathbf{N}^{(e)T}\,dx\,dy \tag{8.201}$$

In what follows we shall assume that the functions p and r are constant over each element and are represented by $p^{(e)}$ and $r^{(e)}$ respectively.

Linear triangular element

We consider a three-node triangular element (e) with nodes ijk as shown in Figure 8.12(a).

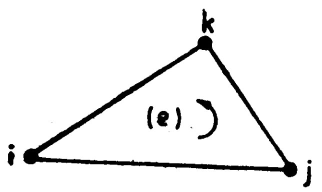

Fig. 8.12(a) Triangular element

Using (8.63), the linear piecewise approximate solution over the element (e) may be written as

$$u^{(e)}=N_iu_i+N_ju_j+N_ku_k=\mathbf{N}^{(e)}\boldsymbol{\phi}^{(e)} \tag{8.202}$$

where

$$\mathbf{N}^{(e)}=[N_iN_jN_k], \qquad \boldsymbol{\phi}^{(e)}=[u_i\ u_j\ u_k]^T$$

$$N_i=\frac{1}{2\Delta^{(e)}}(a_i+b_ix+c_iy)$$

$$N_j=\frac{1}{2\Delta^{(e)}}(a_j+b_jx+c_jy)$$

$$N_k=\frac{1}{2\Delta^{(e)}}(a_k+b_kx+c_ky)$$

$$\Delta^{(e)}=\frac{1}{2}\det\begin{bmatrix}1 & x_i & y_i\\ 1 & x_j & y_j\\ 1 & x_k & y_k\end{bmatrix}$$

$$a_i=x_jy_k-x_ky_j,\ b_i=y_j-y_k,\ c_i=x_k-x_j$$

$$a_j=x_ky_i-x_iy_k,\ b_j=y_k-y_i,\ c_j=x_i-x_k$$

$$a_k=x_iy_j-x_jy_i,\ b_k=y_i-y_j,\ c_k=x_j-x_i$$

Substituting (8.202) into (8.201) and using (8.71)-(8.72) for simplification, we get the element equations (8.200),

$$\mathbf{A}^{(e)}\boldsymbol{\phi}^{(e)}-\mathbf{b}^{(e)}=0 \tag{8.203}$$

where

$$\mathbf{A}^{(e)}=\frac{p^{(e)}}{4\Delta^{(e)}}\begin{bmatrix}b_i^2+c_i^2 & b_ib_j+c_ic_j & b_ib_k+c_ic_k\\ b_ib_j+c_ic_j & b_j^2+c_j^2 & b_jb_k+c_jc_k\\ b_ib_k+c_ic_k & b_jb_k+c_jc_k & b_k^2+c_k^2\end{bmatrix}$$

$$\mathbf{b}^{(e)}=\frac{r^{(e)}\Delta^{(e)}}{3}\begin{bmatrix}1\\1\\1\end{bmatrix},\qquad \boldsymbol{\phi}^{(e)}=\begin{bmatrix}u_i\\u_j\\u_k\end{bmatrix} \tag{8.204}$$

Quadratic triangular element

Next, instead of the linear function (8.202) we use the quadratic approximate function of the form

$$u^{(e)}=N_iu_i+N_lu_l+N_ju_j+N_mu_m+N_ku_k+N_nu_n=\mathbf{N}^{(e)}\boldsymbol{\phi}^{(e)} \tag{8.205}$$

where the six-node triangular element (e) is shown in Figure 8.12(b). The functions N_i, N_l, $\cdots$, N_n may be written with the help of (8.79) as

$$N_i=2L_i^2-L_i,\qquad N_j=2L_j^2-L_j,\qquad N_k=2L_k^2-L_k$$

$$N_l=4L_iL_j,\qquad N_m=4L_jL_k,\qquad N_n=4L_kL_i$$

$$\mathbf{N}^{(e)}=[N_iN_lN_j\cdots N_n]\quad\text{and}\quad\boldsymbol{\phi}^{(e)}=[u_i\ u_l\ u_j\cdots u_n]^T.$$

where

$$\mathbf{A}^{(e)}=\frac{p^{(e)}}{6}\left\{\frac{b}{a}\begin{bmatrix}2 & -2 & -1 & 1\\ -2 & 2 & 1 & -1\\ -1 & 1 & 2 & -2\\ 1 & -1 & -2 & 2\end{bmatrix}+\frac{a}{b}\begin{bmatrix}2 & 1 & -1 & -2\\ 1 & 2 & -2 & -1\\ -1 & -2 & 2 & 1\\ -2 & -1 & 1 & 2\end{bmatrix}\right\}$$

$$\mathbf{b}^{(e)}=abr^{(e)}\begin{bmatrix}1\\1\\1\\1\end{bmatrix},\quad \phi^{(e)}=\begin{bmatrix}u_i\\u_j\\u_k\\u_l\end{bmatrix} \tag{8.210}$$

Triangular element with one curved side

A typical triangular element with one curved side is shown in Figure 8.12(d).

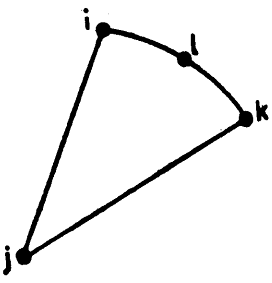

Fig. 8.12(d) Triangular element with one curved side

The approximate solution may be chosen of the form (see (8.131))

$$u^{(e)}=N_iu_i+N_ju_j+N_ku_k+N_lu_l \tag{8.211}$$

where

$$N_i=L_1(1-2L_3),\quad N_j=L_2$$
$$N_k=L_3(1-2L_1),\quad N_l=4L_1L_3$$

The element equations in this case become

$$\mathbf{A}^{(e)}\phi^{(e)}-\mathbf{b}^{(e)}=0 \tag{8.212}$$

where

$$\mathbf{A}^{(e)}=\frac{p^{(e)}}{12\Delta^{(e)}}\times$$

$$\begin{bmatrix}((b_i-b_k)^2+b_k^2) & (b_ib_j-2b_jb_k) & b_ib_k & -4b_k^2\\ & 3b_j^2 & (b_jb_k-2b_kb_j) & (4b_jb_k+4b_jb_i)\\ \text{Symmetric} & & ((b_k-b_i)^2+b_i^2) & -4b_i^2\\ & & & (8b_k^2+8b_ib_k+8b_i^2)\end{bmatrix}$$

$$+\frac{p^{(e)}}{12\Delta^{(e)}}\times\begin{bmatrix} ((c_i-c_k)^2+c_k^2) & (c_ic_j-2c_jc_k) & c_ic_k & -4c_k^2 \\ & 3c_j^2 & (c_ic_k-2c_ic_j) & (4c_jc_k+4c_ic_j) \\ \text{Symmetric} & & ((c_k-c_i)^2+c_i^2) & -4c_i^2 \\ & & & (8c_k^2+8c_ic_k+8c_i^2) \end{bmatrix}$$

$$\mathbf{b}^{(e)}=\frac{\Delta^{(e)}\,r^{(e)}}{6}\begin{bmatrix}1\\2\\1\\2\end{bmatrix} \qquad (8.213)$$

8.7.1 Assembly of element equations

The equation (8.198) becomes

$$\sum_{e=1}^{M}(\mathbf{A}^{(e)}\boldsymbol{\phi}^{(e)}-\mathbf{b}^{(e)})=\mathbf{0} \qquad (8.214)$$

which may be assembled into a single matrix equation

$$\mathbf{A}\boldsymbol{\phi}-\mathbf{b}=\mathbf{0} \qquad (8.215)$$

where

$$\mathbf{A}=\sum_{e=1}^{M}\mathbf{A}^{(e)} \quad \text{is a } N\times N \text{ matrix}$$

and

$$\mathbf{b}=\sum_{e=1}^{M}\mathbf{b}^{(e)} \quad \text{is a } N\times 1 \text{ matrix}$$

The matrix **A** may be arranged into a band matrix. The maximum width of the band is affected by the way the nodal points are numbered.

We now describe the formation of the equation (8.215) when the region $\mathcal{R}$ is composed of only eight triangular elements as shown in Figure 8.13(a).

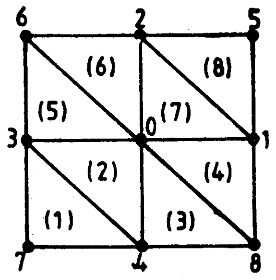

Fig. 8.13(a) Triangular network

By summing the above columns we obtain the elements of the first row of the matrix **A**. The element of the first row of the matrix **b** may also be obtained in a similar manner. The finite element discretization of the differential equation (8.190) at the node '0' is given by the first equation of (8.215).

$$\left(p^{(2)}+\frac{p^{(3)}}{2}+\frac{p^{(4)}}{2}+\frac{p^{(5)}}{2}+\frac{p^{(6)}}{2}+p^{(7)}\right)u_0-\frac{p^{(4)}+p^{(7)}}{2}u_1-\frac{p^{(6)}+p^{(7)}}{2}u_2-\frac{p^{(2)}+p^{(5)}}{2}u_3$$
$$-\frac{p^{(2)}+p^{(3)}}{2}u_4=\frac{h^2}{6}(r^{(2)}+r^{(3)}+r^{(4)}+r^{(5)}+r^{(6)}+r^{(7)}) \qquad (8.218)$$

For the case of constant p and $r(x, y)=0$, the equation (8.190) is the Laplace equation

$$\nabla^2 u=0 \qquad (8.219)$$

and (8.218) becomes

$$4u_0-(u_1+u_2+u_3+u_4)=0 \qquad (8.220)$$

which is the 5-point formula.

Next we determine the equation (8.220) for the rectangular network shown in Figure 8.13(b).

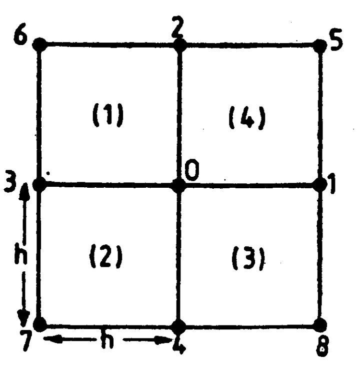

Fig. 8.13(b) Rectangular network

The element equation (8.209) becomes

$$\frac{p^{(e)}}{6}\begin{bmatrix} 4 & -1 & -2 & -1 \\ -1 & 4 & -1 & -2 \\ -2 & -1 & 4 & -1 \\ -1 & -2 & -1 & 4 \end{bmatrix}\begin{bmatrix} u_i \\ u_j \\ u_k \\ u_l \end{bmatrix}=h^2 r^{(e)}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \qquad (8.221)$$

Table 8.6 shows the nodes $ijkl$ of the rectangular element Fig. 8.12(c) which correspond to the nodes of the elements (1)−(4).

TABLE 8.6 ELEMENTS AND NODES

Element	Nodes			
(e)	i	j	k	l
1	3	0	2	6
2	7	4	0	3
3	4	8	1	0
4	0	1	5	2

It is easily verified that the difference equation (8.220) at the node '0' for the Laplace equation (8.219) is given by

$$u_1+u_2+u_3+u_4+u_5+u_6+u_7+u_8-8u_0=0$$

or

$$\delta_x^2 u_{l,m}+\delta_y^2 u_{l,m}+\tfrac{1}{3}\delta_x^2\delta_y^2 u_{l,m}=0 \qquad (8.222)$$

The incorporation of the Dirichlet boundary conditions into the matrix equation (8.215) has already been discussed in Section 8.6.1. The solution of (8.215) after incorporating the boundary conditions gives the approximate solution of (8.190).

8.7.2 Mixed boundary conditions

We consider the differential equation (8.190) subject to the boundary conditions of the form

$$\alpha_1(s)u+\alpha_2(s)\frac{\partial u}{\partial n}=\beta(s) \text{ on } \partial\mathcal{R} \qquad (8.223)$$

where n is the unit outward normal, s is the boundary coordinate measured from a fixed point as shown in Figure 8.14, and $\alpha_1(s)$, $\alpha_2(s)$ and $\beta(s)$ are prescribed functions of s.

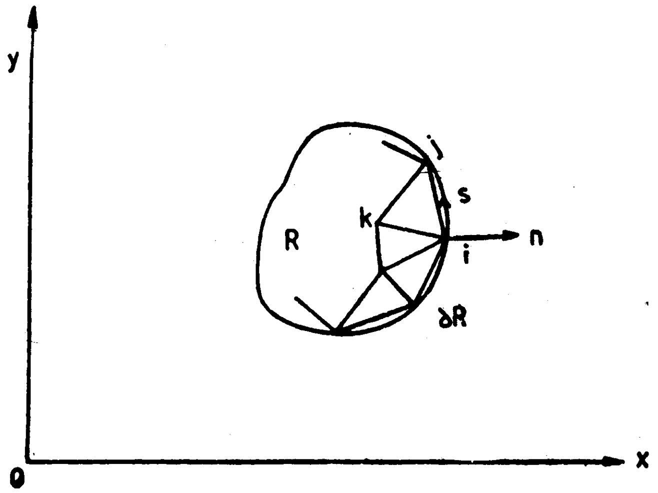

Fig. 8.14 Boundary elements of the domain

Using a similar argument with similar assumptions we may obtain;
For the side jk on the boundary $\partial\mathcal{R}$

$$\mathbf{B}^{(e)} = \frac{p_{jk}(\alpha_1)_{jk}}{(\alpha_2)_{jk}} \frac{s_{kj}}{6} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}$$

$$\mathbf{d}^{(e)} = \frac{p_{jk}\beta_{jk}}{(\alpha_2)_{jk}} \frac{s_{kj}}{2} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \tag{8.233}$$

for the side ki on the boundary $\partial\mathcal{R}$

$$\mathbf{B}^{(e)} = \frac{p_{ki}(\alpha_1)_{ki}}{(\alpha_2)_{ki}} \frac{s_{ik}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

$$\mathbf{d}^{(e)} = \frac{p_{ki}\beta_{ki}}{(\alpha_2)_{ki}} \frac{s_{ik}}{2} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \tag{8.234}$$

If there are two sides of the element (e) on the boundary then the boundary integral becomes a sum for the each side.

Further, if we use a six-node triangular element (e) (Fig. 8.12(b)) and assume that the side ij lies on the boundary then the element matrices $\mathbf{A}^{(e)}$ $\mathbf{B}^{(e)}$, $\mathbf{b}^{(e)}$ and $\mathbf{d}^{(e)}$ are modified. The matrix $\mathbf{d}^{(e)}$ becomes

$$\mathbf{d}^{(e)} = \int_{s_i}^{s_j} \left(\frac{p\beta}{\alpha_2} \begin{bmatrix} L_1(2L_1-1) \\ 4L_1L_2 \\ L_2(2L_2-1) \\ 0 \\ 0 \\ 0 \end{bmatrix} \right) ds$$

$$= \frac{p_{ij}\beta_{ij}s_{ji}}{6(\alpha_2)_{ij}} \begin{bmatrix} 1 \\ 4 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{8.235}$$

Similarly, if we use a rectangulur element (e) (Fig. 8.12(c)) and assume the side jk along the boundary then the matrix $\mathbf{d}^{(e)}$ is given by

$$\mathbf{d}^{(e)}=\int_{s_j}^{s_k}\frac{p\beta}{\alpha_2}\begin{bmatrix}0\\ \frac{1}{2}(1-\eta)\\ \frac{1}{2}(1+\eta)\\ 0\end{bmatrix}ds$$

$$=\frac{2bp_{jk}\beta_{jk}}{(\alpha_2)_{jk}}\begin{bmatrix}0\\ \frac{1}{2}\\ \frac{1}{2}\\ 0\end{bmatrix} \tag{8.236}$$

where $2b$ is the length of the side jk.

The element equation (8.227) can be summed up for all the elements comprising the domain $\mathcal{R}$ and the boundary $\partial\mathcal{R}$ and we obtain the equations of the form (8.215).

8.7.4 Galerkin method

The Galerkin equations (8.161) for the differential equation (8.190) may be written as

$$\iint_{\mathcal{R}^{(e)}} N_i\left[-\frac{\partial}{\partial x}\left(p\frac{\partial u^{(e)}}{\partial x}\right)-\frac{\partial}{\partial y}\left(p\frac{\partial u^{(e)}}{\partial y}\right)-r\right]dx\,dy=0$$

$$i=1,2,\ \ldots,q \tag{8.237}$$

where N_i are the shape functions defined piecewise, element by element and q is the number of unknown nodal quantities assigned to the element (e). We should be careful in choosing $u^{(e)}$ because the second derivative terms in the integrand may become zero inside a finite element. Before substituting for $u^{(e)}$ we express the first and second terms in the integrand in the form

$$\iint_{\mathcal{R}^{(e)}} -N_i\frac{\partial}{\partial x}\left(p\frac{\partial u^{(e)}}{\partial x}\right)dx\,dy$$

$$=-\int_{\partial\mathcal{R}^{(e)}} N_i\,p\frac{\partial u^{(e)}}{\partial x}n_x\,ds+\iint_{\mathcal{R}^{(e)}} p\frac{\partial N_i}{\partial x}\frac{\partial u^{(e)}}{\partial x}dx\,dy$$

TABLE 8.7 PARAMETERS FOR ELEMENT EQUATIONS

Element (e)	Nodes i	j	k
1	2	3	1
2	4	5	2
3	5	6	2
4	6	3	2

Element (e)	Coordinate x_i	x_j	x_k	y_i	y_j	y_k
1	$\frac{1}{2}$	1	1	$\frac{1}{2}$	$\frac{1}{2}$	1
2	0	$\frac{1}{2}$	$\frac{1}{2}$	0	0	$\frac{1}{2}$
3	$\frac{1}{2}$	1	$\frac{1}{2}$	0	0	$\frac{1}{2}$
4	1	1	$\frac{1}{2}$	0	$\frac{1}{2}$	$\frac{1}{2}$

Element (e)	Parameters b_i y_j-y_k	b_j y_k-y_i	b_k y_i-y_j	c_i x_k-x_j	c_j x_i-x_k	c_k x_j-x_i	$\Delta^{(e)}$
1	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1/8
2	$-\frac{1}{2}$	$\frac{1}{2}$	0	0	$-\frac{1}{2}$	$\frac{1}{2}$	1/8
3	$-\frac{1}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	0	$\frac{1}{2}$	1/8
4	0	$\frac{1}{2}$	$-\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	1/8

To assemble all the element equations into a single equation, we may write as

$$\sum_{e=1}^{4} \mathbf{A}^{(e)}\phi^{(e)} = \frac{1}{2}\begin{matrix} u_2 & u_3 & u_1 \\ \left[\begin{matrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{matrix}\right. & & \left.\right] \end{matrix}\begin{bmatrix} u_2 \\ u_3 \\ u_1 \end{bmatrix} + \frac{1}{2}\begin{matrix} u_4 & u_5 & u_2 \\ \left[\begin{matrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{matrix}\right. & & \left.\right] \end{matrix}\begin{bmatrix} u_4 \\ u_5 \\ u_2 \end{bmatrix}$$

$$+\frac{1}{2}\begin{matrix} u_5 & u_6 & u_2 \\ \left[\begin{matrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{matrix}\right. & & \left.\right] \end{matrix}\begin{bmatrix} u_5 \\ u_6 \\ u_2 \end{bmatrix} + \frac{1}{2}\begin{matrix} u_6 & u_3 & u_2 \\ \left[\begin{matrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{matrix}\right. & & \left.\right] \end{matrix}\begin{bmatrix} u_6 \\ u_3 \\ u_2 \end{bmatrix}$$

Using the row-column location of the nodal values u_i, we have

$$
\sum_{e=1}^{4} \mathbf{A}^{(e)}\boldsymbol{\phi}^{(e)} = \tfrac{1}{2} \times
\left[\begin{array}{c|c|c|c|c|c}
u_1 & u_2 & u_3 & u_4 & u_5 & u_6 \\
\hline
1 & 0 & -1 & & & \\
\hline
0 & 1+1+1+1 & -1-1 & 0 & -1-1 & 0+0 \\
\hline
-1 & -1-1 & 2+2 & & & -1 \\
\hline
 & 0 & & 1 & -1 & \\
\hline
 & -1-1 & & -1 & 2+2 & -1 \\
\hline
 & 0+0 & -1 & & -1 & 1+1
\end{array}\right]
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
$$

$$
= \tfrac{1}{2}\begin{bmatrix}
1 & 0 & -1 & 0 & 0 & 0 \\
0 & 4 & -2 & 0 & -2 & 0 \\
-1 & -2 & 4 & 0 & 0 & -1 \\
0 & 0 & 0 & 1 & -1 & 0 \\
0 & -2 & 0 & -1 & 4 & -1 \\
0 & 0 & -1 & 0 & -1 & 2
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
$$

Similarly the right-hand column vector may be assembled as

$$
\sum_{e=1}^{4} \mathbf{b}^{(e)} = \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} u_2 \\ u_3 \\ u_1 \end{matrix}
+ \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} u_4 \\ u_5 \\ u_2 \end{matrix}
+ \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} u_5 \\ u_6 \\ u_2 \end{matrix}
$$

$$
+ \frac{1}{24}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\begin{matrix} u_6 \\ u_3 \\ u_2 \end{matrix}
= \frac{1}{24}\begin{bmatrix} 1+0+0+0 \\ 1+1+1+1 \\ 1+0+0+1 \\ 0+1+0+0 \\ 0+1+1+0 \\ 0+0+1+1 \end{bmatrix}\begin{matrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{matrix}
= \frac{1}{24}\begin{bmatrix} 1 \\ 4 \\ 2 \\ 1 \\ 2 \\ 2 \end{bmatrix}
$$

The assembled matrix incorporating the boundary conditions is given by

$$\begin{bmatrix} 12 & 0 & 2 & 0 & -8 & 0 & -8 & 0 & 0 & 0 \\ 0 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 \\ 2 & 1 & 12 & 0 & 0 & 0 & -8 & 0 & -4 & -4 \\ 0 & 0 & 0 & 16 & -8 & 0 & 0 & 0 & 0 & 0 \\ -8 & 0 & 0 & -8 & 32 & 0 & 0 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 16 & -8 & 0 & -8 & 0 \\ -8 & 0 & -8 & 0 & 0 & -8 & 32 & -8 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8 & 0 & -8 & 32 & 0 & -8 \\ 0 & -4 & -4 & 0 & 0 & -8 & 0 & 0 & 16 & 0 \\ 0 & 0 & -4 & 0 & 0 & 0 & 0 & -8 & 0 & 16 \end{bmatrix} \begin{bmatrix} u_2 \\ u_4 \\ u_5 \\ u_7 \\ u_9 \\ u_{10} \\ u_{11} \\ u_{12} \\ u_{14} \\ u_{15} \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 2 \\ 1 \\ 2 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

Solving we obtain

$u_2 = 0.18159$, $u_4 = 0.29499$, $u_5 = 0.22959$
$u_7 = 0.07172$, $u_9 = 0.11219$, $u_{10} = 0.26374$
$u_{11} = 0.21759$, $u_{12} = 0.13294$, $u_{14} = 0.27864$
$u_{15} = 0.13949$

Example 8.9 Use the finite element method to solve the boundary value problem

$$\nabla^2 u = -12xy, \quad x, y > 0, \quad x^2 + y^2 < 1$$
$$u = 0 \quad \text{on the boundary}$$

with $h = \frac{1}{2}$.

The exact solution of the boundary value problem is

$$u = xy(1 - x^2 - y^2)$$

We discretize the region, $x, y \geqslant 0$, $x^2+y^2 < 1$, with the help of the triangular elements as shown in Figure 8.16(a).

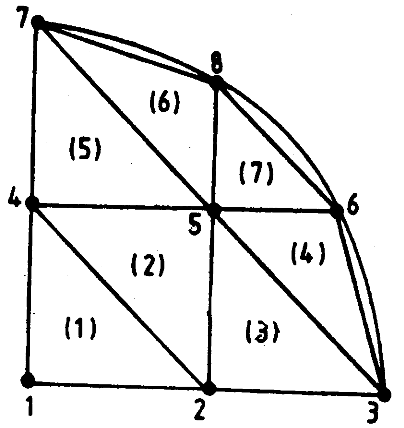

Fig. 8.16(a) Triangular elements

The equation (8.204) gives the element matrix $A^{(e)}$ for a typical triangular element (e) (see Fig. 8.12(a)). The elements of the matrix $\mathbf{b}^{(e)} = [b_i^{(e)}\ b_j^{(e)}\ b_k^{(e)}]^T$ for $r(x, y) = 12xy$ based on exact calculations are

$$b_i^{(e)} = \frac{\Delta^{(e)}}{5}[6x_iy_i + 2x_jy_j + 2x_ky_k + 2(x_iy_j + x_jy_i) + 2(x_iy_k + x_ky_i) + (x_jy_k + y_jx_k)]$$

$$b_j^{(e)} = \frac{\Delta^{(e)}}{5}[2x_iy_i + 6x_jy_j + 2x_ky_k + 2(x_iy_j + x_jy_i) + (x_iy_k + x_ky_i) + 2(x_jy_k + y_jx_k)]$$

$$b_k^{(e)} = \frac{\Delta^{(e)}}{5}[2x_iy_i + 2x_jy_j + 6x_ky_k + (x_iy_j + x_jy_i) + 2(x_iy_k + x_ky_i) + 2(x_jy_k + y_jx_k)]$$

The parameters required for the calculation of the element equations are listed in Table 8.8.

Using the equation (8.204) and Table 8.8 we get the element equations

$$(e) = 1, \begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_4 \end{bmatrix} = \frac{1}{160} \begin{bmatrix} 2 \\ 4 \\ 4 \end{bmatrix},$$

TABLE 8.8 PARAMETERS FOR ELEMENT EQUATIONS

Element (e)	Nodes i	j	k
1	1	2	4
2	4	2	5
3	2	3	5
4	5	3	6
5	4	5	7
6	7	5	8
7	5	6	8

Element	Coordinates					
(e)	x_i	x_j	x_k	y_i	y_j	y_k
1	0	1/2	0	0	0	1/2
2	0	1/2	1/2	1/2	0	1/2
3	1/2	1	1/2	0	0	1/2
4	1/2	1	$\sqrt{3}/2$	1/2	0	1/2
5	0	1/2	0	1/2	1/2	1
6	0	1/2	1/2	1	1/2	$\sqrt{3}/2$
7	1/2	$\sqrt{3}/2$	1/2	1/2	1/2	$\sqrt{3}/2$

Element	Parameters						
(e)	b_i y_j-y_k	b_j y_k-y_i	b_k y_i-y_j	c_i x_k-x_j	c_j x_i-x_k	c_k x_j-x_i	$\Delta^{(e)}$
1	−1/2	1/2	0	−1/2	0	1/2	1/8
2	−1/2	0	1/2	0	−1/2	1/2	1/8
3	−1/2	1/2	0	−1/2	0	1/2	1/8
4	−1/2	0	1/2	$\left(\frac{\sqrt{3}}{2}-1\right)$	$\frac{1}{2}(1-\sqrt{3})$	1/2	$\frac{1}{8}(\sqrt{3}-1)$
5	−1/2	1/2	0	−1/2	0	1/2	1/8
6	$\frac{1}{2}(1-\sqrt{3})$	$\frac{1}{2}(\sqrt{3}-2)$	1/2	0	−1/2	1/2	$\frac{1}{8}(\sqrt{3}-1)$
7	$\frac{1}{2}(1-\sqrt{3})$	$\frac{1}{2}(\sqrt{3}-1)$	0	$\frac{1}{2}(1-\sqrt{3})$	0	$\frac{1}{2}(\sqrt{3}-1)$	$\frac{1}{8}(\sqrt{3}-1)^2$

$$(e)=2,\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} u_4 \\ u_2 \\ u_5 \end{bmatrix}=\frac{1}{160}\begin{bmatrix} 14 \\ 14 \\ 22 \end{bmatrix}$$

$$(e)=3,\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} u_2 \\ u_3 \\ u_5 \end{bmatrix}=\frac{1}{160}\begin{bmatrix} 12 \\ 14 \\ 24 \end{bmatrix}$$

$(e)=4,$

$$\begin{bmatrix} 2(\sqrt{3}-1) & 2-\sqrt{3} & -\sqrt{3} \\ 2-\sqrt{3} & \sqrt{3}-1 & -1 \\ -\sqrt{3} & -1 & \frac{1}{4}(\sqrt{3}+1) \end{bmatrix} \begin{bmatrix} u_5 \\ u_3 \\ u_6 \end{bmatrix} = \frac{(\sqrt{3}-1)}{160} \begin{bmatrix} 28+8\sqrt{3} \\ 22+6\sqrt{3} \\ 20+16\sqrt{3} \end{bmatrix}$$

$(e)=5,$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_4 \\ u_5 \\ u_7 \end{bmatrix} = \frac{1}{160} \begin{bmatrix} 12 \\ 24 \\ 14 \end{bmatrix}$$

$(e)=6,$

$$\begin{bmatrix} \sqrt{3}-1 & 2-\sqrt{3} & -1 \\ 2-\sqrt{3} & 2(\sqrt{3}-1) & -\sqrt{3} \\ -1 & -\sqrt{3} & (\sqrt{3}+1) \end{bmatrix} \begin{bmatrix} u_7 \\ u_5 \\ u_8 \end{bmatrix} = \frac{(\sqrt{3}-1)}{160} \begin{bmatrix} 22+6\sqrt{3} \\ 28+8\sqrt{3} \\ 20+16\sqrt{3} \end{bmatrix}$$

$(e)=7,$

$$\begin{bmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_5 \\ u_6 \\ u_8 \end{bmatrix} = \frac{(\sqrt{3}-1)^2}{160} \begin{bmatrix} 28+16\sqrt{3} \\ 26+22\sqrt{3} \\ 26+22\sqrt{3} \end{bmatrix}$$

Assembling the element equations we obtain

$$\begin{bmatrix} 2 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -2 & 0 & 0 & 0 \\ 0 & -1 & \sqrt{3} & 0 & 2-\sqrt{3} & -1 & 0 & 0 \\ -1 & 0 & 0 & 4 & -2 & 0 & -1 & 0 \\ 0 & -2 & 2-\sqrt{3} & -2 & 2+4\sqrt{3} & -\sqrt{3}-1 & 2-\sqrt{3} & -\sqrt{3}-1 \\ 0 & 0 & -1 & 0 & -\sqrt{3}-1 & \frac{5}{4}+\frac{\sqrt{3}}{4} & 0 & 0 \\ 0 & 0 & 0 & -1 & 2-\sqrt{3} & 0 & \sqrt{3} & -1 \\ 0 & 0 & 0 & 0 & -\sqrt{3}-1 & 0 & -1 & \sqrt{3}+2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \\ u_7 \\ u_8 \end{bmatrix}$$

$$
= \frac{1}{160}\begin{bmatrix} 2 \\ 30 \\ 10+16\sqrt{3} \\ 32 \\ 78+48\sqrt{3} \\ 40\sqrt{3} \\ 10+16\sqrt{3} \\ 40\sqrt{3} \end{bmatrix}
$$

The boundary conditions give

$$u_1 = u_2 = u_3 = u_6 = u_8 = u_7 = u_4 = 0$$

Incorporating the boundary conditions we get

$$u_5 = (39+24\sqrt{3})/160(1+2\sqrt{3}) = 0.1128$$

The exact value is given by

$$u_5 = 0.125$$

Next, we take the elements (4), (6) and (7) as the triangular elements with one curved side (see Fig. 8.16(b)).

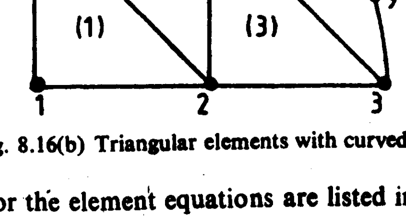

Fig. 8.16(b) Triangular elements with curved sides

The parameters for the element equations are listed in Table 8.9.
We also have

$$\Delta^{(4)} = \Delta^{(6)} = \frac{\pi}{12} + \frac{\sqrt{3}}{8} - \frac{3}{8}$$

$$\Delta^{(7)} = \frac{\pi}{12} - \frac{\sqrt{3}}{4} + \frac{1}{4}$$